

THE CUP SUBALGEBRA HAS THE ABSORBING AMENABILITY PROPERTY

BY ARNAUD BROTHIER¹ AND CHENXU WEN²

ABSTRACT. Consider an inclusion of diffuse von Neumann algebras $A \subset M$. We say that $A \subset M$ has the absorbing amenability property if for any diffuse subalgebra $B \subset A$ and any amenable intermediate algebra $B \subset D \subset M$ we have that D is contained in A . We prove that the cup subalgebra associated to any subfactor planar algebra has the absorbing amenability property.

INTRODUCTION AND MAIN RESULTS

Amenability is a fundamental concept in various area of mathematics. Connes proved the striking result that a von Neumann algebra is amenable if and only if it is hyperfinite [Con76]. In this article, we study maximal amenable subalgebras. Fuglede and Kadison showed that any II_1 factor contains a maximal amenable subfactor [FK51]. Popa exhibited the first example of an abelian maximal amenable subalgebra of a II_1 factor, thus giving a counter-example to a question of Kadison [Pop83]. He defined the notion of asymptotic orthogonality property (AOP) and showed that a singular maximal abelian subalgebra (masa) with the AOP is maximal amenable. Many other examples have been given using the same strategy [Ge96, She06, CFRW10, Bro14, Hou14a, BCa]. A completely new strategy in proving maximal amenability has been given in [BCb].

Peterson conjectured that any maximal amenable subalgebra of a free group factor is the unique maximal amenable extension of any of its diffuse subalgebra. Inspired by this question and the work of Houdayer on maximal Gamma extensions [Hou14b, Hou15], we consider the notion of absorbing amenability property (AAP). An inclusion of von Neumann algebras $A \subset M$ has the AAP if for any diffuse subalgebra $B \subset A$ and any amenable intermediate algebra $B \subset D \subset M$ we have that D is contained in A . In particular, if A is amenable, then it is maximal amenable. Houdayer proved that the generator masa has the AAP [Hou15]. The second author showed that the radial masa has the AAP [Wen].

In this article, we present a new class of examples that have the AAP. Those examples are constructed with Jones planar algebras [Jon]. If \mathcal{P} is a subfactor planar algebra, then we can associate to it a II_1 factor M [GJS10]. This II_1 factor is isomorphic to an interpolated free group factor $L(\mathbb{F}_t)$ where t is a linear combination of the index and the global index of \mathcal{P} [Dyk94, Rad94, GJS11, Har13]. This factor admits a generic abelian subalgebra $A \subset M$ that we call the cup subalgebra. The first author previously proved that the cup subalgebra is maximal amenable [Bro14]. We prove the following theorem:

Theorem A. *The cup subalgebra associated to any subfactor planar algebra has the absorbing amenability property.*

This provides many examples of subalgebras of interpolated free group factors with the AAP. Note, it is still unknown if there exists a subfactor planar algebra such that its associated cup subalgebra is isomorphic to the generator or the radial masa.

¹Vanderbilt University, Department of Mathematics, 1326 Stevenson Center Nashville, TN, 37240, USA, arnaud.brothier@vanderbilt.edu

²Vanderbilt University, Department of Mathematics, 1326 Stevenson Center Nashville, TN, 37240, USA, chenxu.wen@vanderbilt.edu

Acknowledgement. We express our gratitude to Cyril Houdayer and Jesse Peterson for encouragements and making comments on an earlier version of this manuscript.

1. PRELIMINARIES

1.1. Planar algebras. A planar algebra is a collection of complex $*$ -algebras $\mathcal{P} = (\mathcal{P}_n^\pm : n \geq 0)$ on which the set of shaded planar tangles acts. See [Jon, Jon12] for more details. We follow similar conventions that was used in [CJS14] for drawing a shaded planar tangle. We decorate strings with natural numbers to indicate that they represent a given number of parallel strings. The distinguished interval of a box is decorated by a dollar sign if it is not at the top left corner. We do not draw the outside box and will omit unnecessary decorations. The left and right traces of a planar algebra are the maps $\tau_l : \mathcal{P}_n^\pm \rightarrow \mathcal{P}_0^\mp$ and $\tau_r : \mathcal{P}_n^\pm \rightarrow \mathcal{P}_0^\pm$ defined for any $n \geq 0$ such that

$$\tau_l(x) = \left(\begin{array}{c} \boxed{x} \end{array} \right) \text{ and } \tau_r(x) = \left(\begin{array}{c} \boxed{x} \end{array} \right) \text{ for any } x \in \mathcal{P}_n^\pm.$$

Suppose that $\mathcal{P}_0^\pm = \mathbb{C}$. The planar algebra is called spherical if the two traces agree on each element of \mathcal{P} . We say that \mathcal{P} is non-degenerate if the sesquilinear forms $(x, y) \mapsto \tau_l(xy^*)$ and $(x, y) \mapsto \tau_r(xy^*)$ are positive definite. A subfactor planar algebra is a planar algebra such that each space \mathcal{P}_n^\pm is finite dimensional, $\mathcal{P}_0^\pm = \mathbb{C}$, \mathcal{P} is spherical and non-degenerate. The modulus of a subfactor planar algebra is the value of a closed loop.

1.2. Construction of a II_1 factor. We recall a construction due to Jones et al. [JSW10]. Consider the direct sum $Gr\mathcal{P} = \bigoplus_{n \geq 0} \mathcal{P}_n^+$ that we equipped with the following Bacher product and involution:

$$xy = \sum_{a=0}^{\min(2n, 2m)} \left(\begin{array}{c} \boxed{x} \quad \boxed{y} \end{array} \right)^a, \text{ and } x^\dagger = - \left(\begin{array}{c} \boxed{x^*} \end{array} \right)^{\$}, \text{ where } x \in \mathcal{P}_n^+ \text{ and } y \in \mathcal{P}_m^+.$$

Consider the linear form $\tau : Gr\mathcal{P} \rightarrow \mathbb{C}$ that sends $x \in \mathcal{P}_0^+$ to itself and 0 to any element in \mathcal{P}_n^+ if $n \neq 0$. The vector space $Gr\mathcal{P}$ endowed with those operation is an associative $*$ -algebra with a faithful tracial state. Let H be the completion of $Gr\mathcal{P}$ for the inner product $(x, y) \mapsto \tau(xy^*)$. The left multiplication of $Gr\mathcal{P}$ on H is bounded and defines a $*$ -representation [GJS10, JSW10]. Let M be the von Neumann algebra generated by $Gr\mathcal{P}$ inside $B(H)$. It is an interpolated free group factor [GJS11, Har13]. We define another multiplication on $Gr\mathcal{P}$ by requiring that if $x \in \mathcal{P}_n$ and $y \in \mathcal{P}_m$, then

$$x \bullet y = \left(\begin{array}{c} \boxed{x} \quad \boxed{y} \end{array} \right) \in \mathcal{P}_{n+m}^+.$$

Denote by $x^{\bullet n}$ the n -th power of x for this multiplication. Remark, $\|a \bullet b\|_2 = \|a\|_2 \|b\|_2$, for all $a \in \mathcal{P}_n$ and $b \in \mathcal{P}_m$. Therefore, this multiplication is a continuous bilinear form for the L^2 -norm $\|\cdot\|_2$ of M . We extend this operation on $L^2(M) \times L^2(M)$ and still denote it by \bullet .

1.3. The cup subalgebra. Let \cup be the unity of the $*$ -algebra \mathcal{P}_1^+ , viewed as an element of M [GJS10]. Let $A \subset M$ be the von Neumann subalgebra generated by \cup . We call it the cup subalgebra.

1.4. Strong asymptotic orthogonal property. Popa introduced the notion of asymptotic orthogonality property (AOP) in [Pop83]. We consider a strengthening of this notion which was used by Houdayer and the second author [Hou14b, Wen].

Definition 1.1. Let $A \subset M$ be a diffuse subalgebra of a tracial von Neumann algebra. This inclusion has the strong asymptotic orthogonality property (SAOP) if for any free ultrafilter ω and any diffuse subalgebra $B \subset A$ we have

$$xy \perp yx \text{ for any } x \in (M^\omega \cap B') \ominus A^\omega \text{ and } y \in M \ominus A.$$

Note, a diffuse subalgebra $A \subset M$ has the SAOP if and only if it has the AOP relative to all of its diffuse subalgebras in the sense of Houdayer [Hou14b, Definition 5.1].

The following theorem is an extension of a theorem of Popa [Pop83].

Theorem 1.2. [Hou14b, Theorem 8.1] *If $A \subset M$ is a diffuse subalgebra with the SAOP such that $L^2(M) \ominus L^2(A)$ is a mixing A -bimodule (e.g. is a direct sum of the coarse bimodule $L^2(A) \otimes L^2(A)$), then it has the AAP.*

See also [Wen, Proposition 2.1].

2. PROOF OF THE MAIN THEOREM

Proposition 2.1. *Let (A, τ) be a tracial von Neumann algebra and $B \subset A$ a diffuse subalgebra. Denote by $L^2(A)$ the Gelfand-Naimark-Segal completion of A for the trace τ . Consider a sequence $\xi = (\xi_n : n \geq 0)$ of unit vectors of the coarse bimodule $L^2(A) \otimes L^2(A)$. Suppose that for any $b \in B$ we have $\lim_{n \rightarrow \infty} \|b \cdot \xi_n - \xi_n \cdot b\| = 0$. Then, if $p \in B(L^2(A))$ is a finite rank projection, then $\lim_{n \rightarrow \infty} \|(p \otimes 1)\xi_n\| = \lim_{n \rightarrow \infty} \|(1 \otimes p)\xi_n\| = 0$.*

Proof. Let A, B, ξ , and p as above. It is sufficient to prove the proposition when p is a rank one projection. Let $\eta \in L^2(A)$ be a unit vector such that $p = p_\eta$ is the rank one projection onto $\mathbb{C}\eta$. Consider $0 < \varepsilon < 1$ and a natural number I such that $16/(I+1) < \varepsilon$. Since B is diffuse, there exists a sequence of unitaries $(u_n)_n$ in B such that $\lim_{n \rightarrow \infty} \langle u_n \cdot \zeta_1, \zeta_2 \rangle = 0$ for any $\zeta_1, \zeta_2 \in L^2(A)$. Consider the quantity $\delta = \max(|\langle u_n \cdot \eta, u_m \cdot \eta \rangle| : n \neq m, n, m \leq I)$. By [Hou14a, Proposition 2.3], we have that

$$\sum_{i=0}^I \|(p_{u_i \cdot \eta} \otimes 1)\xi_n\|^2 \leq g(\delta) \|\xi_n\|^2 \text{ for any } n \geq 0,$$

where g is a positive function satisfying $\lim_{\delta \rightarrow 0} g(\delta) = 1$. Hence, there exists a subsequence $(v_n)_n$ such that

$$\sum_{i=0}^I \|(p_{v_i \cdot \eta} \otimes 1)\xi_n\|^2 \leq 2\|\xi_n\|^2 = 2 \text{ for any } n \geq 0.$$

Let $\lambda : B \rightarrow B(L^2(A) \otimes L^2(A))$ be the left action of B on the coarse bimodule $L^2(A) \otimes L^2(A)$. Observe, $p_{v_i \cdot \eta} \otimes 1 = \lambda(v_i) \circ (p_\eta \otimes 1) \circ \lambda(v_i)^*$ and v_i is a unitary, for any $i \geq 0$. Therefore, $\|(p_{v_i \cdot \eta} \otimes 1)\xi_n\| = \|(p_\eta \otimes 1)v_i^* \cdot \xi_n\|$ for any $n, i \geq 0$. By assumption, there exists $N > 0$ such that for any $n \geq N$ and $i \leq I$ we have $\|v_i^* \cdot \xi_n - \xi_n \cdot v_i^*\| < \varepsilon/4$. Therefore,

$$\begin{aligned} \|(p_\eta \otimes 1)\xi_n\| &= \|(p_\eta \otimes 1)(\xi_n \cdot v_i^*)\| \\ &\leq \|(p_\eta \otimes 1)(v_i^* \cdot \xi_n - \xi_n \cdot v_i^*)\| + \|(p_\eta \otimes 1)(v_i^* \cdot \xi_n)\| \\ &\leq \varepsilon/4 + \|(p_{v_i \cdot \eta} \otimes 1)\xi_n\| \text{ for any } n \geq N, i \leq I. \end{aligned}$$

We obtain

$$\begin{aligned} \sum_{i=0}^I \|(p_\eta \otimes 1)\xi_n\|^2 &\leq \sum_{i=0}^I (\varepsilon^2/16 + \varepsilon/2 \|(p_{v_i \cdot \eta} \otimes 1)\xi_n\| + \|(p_{v_i \cdot \eta} \otimes 1)\xi_n\|^2) \\ &\leq (I+1)(\varepsilon/16 + \varepsilon/2) + 2 \text{ for any } n \geq N. \end{aligned}$$

Therefore, $\|(p_\eta \otimes 1)\xi_n\|^2 \leq \varepsilon/16 + \varepsilon/2 + 2\varepsilon/16 \leq \varepsilon$ for any $n \geq N$. The same proof shows that there exists $M > 0$ such that for any $n \geq M$ we have $\|(1 \otimes p_\eta)\xi_n\|^2 \leq \varepsilon$. This proves the proposition. \square

Fix a subfactor planar algebra \mathcal{P} with modulus $\delta > 1$ and denote by $A \subset M$ its associated cup subalgebra. Consider the subspace $V_n \subset \mathcal{P}_n^+, n \geq 0$ of elements that vanishes when they are capped off on the top left corner and vanished when they are capped off on the top right corner.

Let $V \subset L^2(M)$ be their orthogonal direct sum. By [JSW10, Theorem 4.9], the following map is an isomorphism of A -bimodules:

$$\phi : L^2(A) \oplus (L^2(A) \otimes V \otimes L^2(A)) \longrightarrow L^2(M), a + b \otimes v \otimes c \longmapsto a + b \bullet v \bullet c.$$

This implies that the A -bimodule $L^2(M) \ominus L^2(A)$ is isomorphic to an infinite direct sum of the coarse bimodule. We identify $L^2(M)$ with $\phi^{-1}(L^2(M))$.

Consider the finite dimensional subspace $L_m = \text{Span}(\cup^{\bullet k} : k \leq m) \subset A$ for $m \geq 0$, where $\cup^{\bullet 0} = 1 \in \mathcal{P}_0^+$. Denote by L_m^\perp the orthogonal complement of L_m inside $L^2(A)$ for any $m \geq 0$.

Lemma 2.2. *Let $m \geq 0$ and $x \in M \cap L_m^\perp \otimes V \otimes L_m^\perp$, $y \in M \cap L_m \otimes V \otimes L_m$. Then $xy \in L_m^\perp \otimes V \otimes L_m$ and $yx \in L_m \otimes V \otimes L_m^\perp$. In particular, $xy \perp yx$.*

Proof. Consider $x = \cup^{\bullet k} \bullet v \bullet \cup^{\bullet l}$ and $y = \cup^{\bullet s} \bullet w \bullet \cup^{\bullet t}$, where $s, t < m+1 \leq k, l$ and $v, w \in V \cap \text{Gr}\mathcal{P}$. We have that

$$xy = \sum_{i=0}^{s+1} \delta^{[i/2]} \cup^{\bullet k} \bullet v \bullet \cup^{\bullet(l+s-i)} \bullet w \bullet \cup^{\bullet t},$$

where $[i/2] = i/2$ if i is even and $i/2 - 1/2$ if i is odd. Observe, L_m^\perp is equal to the closure of $\text{Span}(\cup^{\bullet k} : k \geq m+1)$. Therefore, $xy \in L_m^\perp \otimes V \otimes L_m$ and similarly $yx \in L_m \otimes V \otimes L_m^\perp$. The space $M \cap L_m^\perp \otimes V \otimes L_m$ (resp. $M \cap L_m \otimes V \otimes L_m^\perp$) is the weak closure of $\text{Span}(\cup^{\bullet k} \bullet v \bullet \cup^{\bullet l} : k, l \geq m+1, v \in V \cap \text{Gr}\mathcal{P})$ (resp. $\text{Span}(\cup^{\bullet s} \bullet w \bullet \cup^{\bullet t} : s, t \leq m, w \in V \cap \text{Gr}\mathcal{P})$). This concludes the proof by a density argument. \square

We are ready to prove the main theorem of the article.

Proof of Theorem A. Let \mathcal{P} be a subfactor planar algebra, $A \subset M$ its associated cup subalgebra, and $B \subset A$ a diffuse subalgebra. Consider $x \in M^\omega \ominus A^\omega$ in the relative commutant of B and $y \in M \ominus A$, where ω is a free ultrafilter on \mathbb{N} . Let us show that $xy \perp yx$. Observe, $\text{Gr}\mathcal{P}$ is a weakly dense $*$ -subalgebra of M . Therefore, we can assume that $y \in \text{Gr}\mathcal{P}$ by Kaplansky density theorem. This implies that there exists $m \geq 0$ such that $y \in \text{Gr}\mathcal{P} \cap L_m \otimes V \otimes L_m$. Let $(x_n)_n$ be a representative of x in the ultrapower M^ω . We can assume that for any $n \geq 0$ we have $x_n \in L^2(M) \ominus L^2(A)$. Let $p \in B(L^2(A))$ be the orthogonal projection onto L_m . It is a finite rank projection. Therefore, by Proposition 2.1, $(p \otimes 1)x = (1 \otimes p)x = 0$. Hence, we can assume that $x_n \in L_m^\perp \otimes V \otimes L_m^\perp$ for any $n \geq 0$. Lemma 2.2 implies that $x_n y \perp y x_n$ for any $n \geq 0$. This implies that $xy \perp yx$.

Theorem 1.2 implies that $A \subset M$ has the AAP. \square

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